Math Review Lab Guide — Experiment A01

Welcome to Astronomy!

This first lab is dedicated to a quick math review. I would be willing to bet most of you are wondering why you need such a simple math review. Well, we have seen time and time again that the number one cause of students losing points on their lab reports is a simple math mistake – MOST of the time coming from the misuse of a calculator! In this introductory lab, you will complete a series of math review problems and then turn in **individual** lab reports (which, if you paid attention to the syllabus, is **not** the usual procedure in this lab). This is also a good opportunity to practice the proper way of filling out and submitting your lab reports on *Blackboard*. Also note that a general math review is given as part of the lab manuals *Appendix*, which should have been provided to you through *Blackboard* or email. This lab guide should be used as a completion aid; something you can read while you complete the lab in order to fully develop the skills and help you in difficult spots.

Order of Operations

I'm sure many of you have been on an internet fight concerning the "correct" solution to some long expression of products, sums, etc. I am positive many of you have used PEMDAS in order to solve these problems, and you would be correct! The only issue is that the *order of operations* is a procedure that is an *agreed upon hierarchical order*, not necessarily something that pops right out after considering the fundamentals of mathematics. What I mean is that mathematics itself is not necessarily dependent on this order. Our answers are dependent since they CAN change depending on the order, but it's like (theoretically) swapping the tall support structures in a tall building with one another – the rest of the structure doesn't care as long as they're there in the first place!

What does this mean for you? Not much, this is more a rant to promote the exhaustive use of parenthesis so as to not confuse anyone (parentheses can still be considered a construct, but it is well agreed that grouping terms in parenthesis always signifies that the operations inside be particularly distinguished from those outside).

HOWEVER, this is not to diminish the importance of PEMDAS; *Please Excuse My Dear Aunt Sally; Parentheses, Exponents, Multiplication, Division, Addition, and finally Subtraction.* This is pretty self-explanatory; work out anything in parenthesis first (while simultaneously also applying PEMDAS inside them as well), work out any exponents, multiply, divide, add, and then subtract. Then, if you have multiple of the same procedures in a row, you just work left to right.
For example, in problem 1a.), the answer is NOT zero. First, we calculate 5/8, and then 2+3 separately. We then subtract 5/8 from 2+3 giving us 4.375. A cool thing would be checking your calculator – whether it be an app on your phone or a designated machine. What do you get when you put this expression in as it is written (without parenthesis)? Did it give you the correct answer

(considering PEMDAS) or did it give you zero? This is a good check whether our calculator is programed to consider the order of operations or not. HOWEVER, if it *does*, do not put all your faith into it – it is always a good idea to use parentheses wherever possible in order to mitigate silly mistakes, especially when we start getting into more difficult expressions.

Checking Your Calculator

This question is not only a good way to check that you at least have intermediate calculator skills, but it also allows us to take a glimpse at some astronomy principles in action!

First off, Kepler's third law is pretty interesting – it is telling you that the period of a planets revolution about its star is related very close to its *semi-major axis (a,* which is just the longest "radius" of an ellipse – which we know is the shape of all planets orbits!). Something cool happens, though, when you express *P* in terms of Earth years and *a* in terms of *astronomical units (AUs)* (the mean (average) distance from the center of the Earth to the center of our Sun; 149.6 million km). If you do this, then the proportionality constant between the **square of the period** and the **cube of the semi-major axis** is equal to one! This is pretty astounding if you think about it – this is the same as saying that in these units the period squared is EQUAL to the semimajor axis cubed. Two properties of a planets orbit related in such a beautiful way! Now this isn't magic though; this relation can be derived from Newtons laws of gravity, not just one found by looking at data from planets we observe orbiting the sun. This will be saved for a later discussion, however.

The important part for now is to show that you can plug in the right values and get the result that k=1. So, in your calculator you should be getting in the habit of using parenthesis ANYWHERE and EVERYWHERE. For example, using the values given to you:

$$(\,(11.862)\text{^}(2)\,)\;\div\;(\,(5.2)\text{^}(3)\,)\;=\;\_\_$$

You should see this is VERY close to one, so close to one in fact that we'll say its equal to one. Parentheses are your friend, and you'll see that using them will be important when we get to more difficult problems.

Scientific Notation

Scientific notation is just a way of presenting a number in a clean manner AND makes such numbers easy to manipulate algebraically. This is especially useful for very big or very small numbers. How this is done is by making use of *exponents,* and more specifically the number 10 raised to some number. This is useful because in our number system, multiplying (or dividing) some number by 10 will shift the decimal place to the right or left. Let us start by laying the groundwork on how you may think of exponents and scientific notation in general.

Basics of Exponents

First off is the obvious: what does an exponent represent? All it does is tell you how many times you should multiply a specific number by itself. For example:

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10$$

Well, this is all fine, but what about a negative exponent? A negative exponent tells you that you should do the above operation in the *denominator, i.e.,* when you see a negative exponent you should think:

$$10^{-5} = \frac{1}{10^5} = \frac{1}{(10 \times 10 \times 10 \times 10 \times 10)}$$

So, what are some of the rules regarding exponents?

1.) When a number that is already raised to an exponent is raised to another exponent, it is equivalent to the same number raised to the product of the two exponents.

$$(2^3)^2 = 2^{3\times 2}$$

2.) When you multiply two of the same numbers that are each raised to some exponent, it is equivalent to the number raised to the sum of the two exponents.

$$(4^2) \times (4^3) = (4 \times 4) \times (4 \times 4 \times 4) = (4)^{2+3}$$

Note that this can be applied to negative exponents:

$$(4^{-2}) \times (4^3) = \left(\frac{1}{4 \times 4}\right) \times (4 \times 4 \times 4) = (4)^{-2+3}$$

Also note that rule #1 can be derived from this rule.

This all makes sense, but what about fractional exponents? Well, these are just *roots*. For example, we are aware of the *square-root* operator. This can be written in terms of an exponent as:

$$\sqrt{4} = 4^{1/2} = 2$$

This square-root (or radical) is just telling you to find the number that, when multiplied by itself, results in the number underneath it. We can then expand this idea to other roots, for example the *cubed root* (the number that when multiplied by itself three times equals the number under the radical) is:

$$\sqrt[3]{27} = 27^{1/3} = 3$$

How is it that fractional exponents are equivalent to finding the roots? The fractional exponent is essentially the inverse operation of a regular exponent. Let's figure out why. If we square a number and then take the square root, then it should be clear that we get the same number back again, right?

$$\sqrt{(5)^2} = 5$$

Now let's apply rule 2.) above with the assumption that a radical can be thought of as an exponent (but here we are not sure if it is a fraction yet). In order for the same number to pop back out, the exponent must be one (because any number raised to the exponent 'one' is itself). I'll use 'δ' to symbolize the exponential form of the radical and then we will solve for it:

$$((5)^2)^\delta = (5)^{2\times\delta} = (5)^1$$

In order for this to work out, δ must be ½, since 2 x ½ =1. There it is! Root operators are just fractional exponents.

<u>Lab Problems 3 and 4</u>

In the first table in your lab manual, you see that we have 10 raised to a variety of different numbers. To truly grasp the idea of scientific notation, lets multiply each of these by one. Now, of course this doesn't actually change anything (as it shouldn't; we wouldn't want that), but it will help us develop a method of understanding these numbers. For the first row, we have now:

$$1 \times 10^1 = 10$$

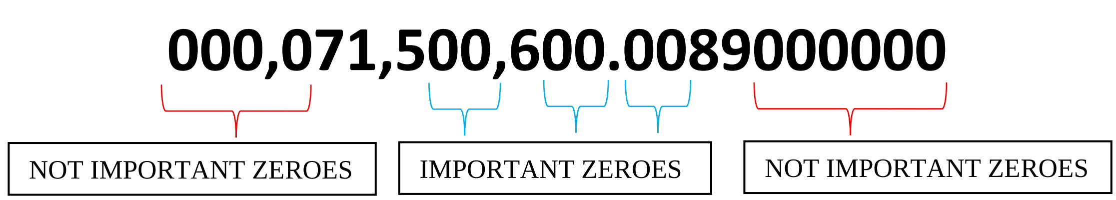
Now we also must establish that when we write a number like '1' or '234', we are assuming that there is a decimal at the end, i.e., '1.0' and '234.0' respectively. So above we now have:

$$1.0 \times 10^1 = 10.0$$

We can now establish a rule in order to turn a number written in scientific notation to one in "regular" form (its full decimal form): the number that the 10 is raised to is the number of decimal places we move to the **RIGHT (for a positive exponent)** or to the **LEFT (for a negative exponent)**. Since the 10 is raised to 'one' above, we can imagine the decimal "hopping" over one place to the right. Now let's try the second row; here the exponent is a two. We can imagine "hopping" over 2 places, giving us 100.0! Of course, for negative exponents it's just as easy, take row 'd' for example. As we said above, we now move two places to the LEFT, and we get 0.01! Now 10 to the zeroth power is simple – what happened when you don't move the decimal at all?

Now for the second table you are considering values other that '1' times 10 to some power. You are still doing the same thing, taking the decimal in the number present before the 10 and shifting it a number of places depending on the value of the exponent! See, simple!

For the third and fourth table, let's consider a more difficult example that you can apply to all of these. What is the number 34,007,000 written in scientific notation? Well, there are plenty of ways we can write it, but let's agree that, unless told differently, when you write something in scientific notation you will only keep one number in front of (i.e., to the left of) your decimal. What does this mean? Well, in this example we are looking for "some number" times 10 to "some exponent." This "some number" is easy to find now that we have established that we want only one digit in front of the decimal; 3.4007000! Now, notice how we have 3 zeros after the seven to the right of the decimal. This is equivalent (logically) to writing the number '5' like '0005.' These extra zeros are just unnecessary. HOWEVER, see that the two zeros between the 4 and 7 ARE important! This importance is obvious by seeing that while before '0005' is numerically equivalent to '5' (thus we can drop the zeros), '1004' doesn't equal '14'!



Continuing with the example; now we need to figure out the "some exponent." Here, let's take the original number (34,007,000.0 – the zero here IS important to define the decimal location.

Later we will touch on how an additional zero can tell us how *accurate* a number is) and see how many decimal places we "hopped" to get "some number." In this case, we moved the decimal seven places to the left (See that this becomes a POSITIVE exponent since we are taking the rules we established before and going in reverse! It is also a good sanity check to note that 34,007,000 is a BIG number, and negative exponents are related to SMALL number, so having a POSITIVE exponent should already be in your head). Thus, in scientific notation:

$$34{,}007{,}000 \ = \ 3.4007x10^7$$

Now you can apply this method to finish up these tables!

For the fifth table (question number four if you've been following along), we are now doing operations on separate numbers that are presented in scientific notation. It may look daunting, but if you convince yourself that a number in scientific notion is just a *number* like any other, then the pieces will come together quickly. What these problems boil down to is an exercise on whether or not you remember the rules for addition/subtraction and multiplication/division. I can tell you're rolling your eyes right now but trust me a quick tutorial on applying those rules here may be beneficial.

This first one (a.) is easy; it uses the *associative property of multiplication.* In this case you have a string of factors all multiplied by each other. The best way to think of this is by regrouping the non-exponential terms together:

$$(A) \times (B \times 10^4) \ = \ (A \times B) \times 10^4 \ = \ C \times 10^4$$

Where C=AxB. Part (d.) is similar in that now we can group the exponents together and use the rules I laid out above:

$$(A \times 10^4) \times (B \times 10^5) \ = \ (A \times B) \times (10^4 \times 10^5) = \ C \times 10^9$$

Now (b.) is just as simple in theory. You are just adding two numbers together, and with the order of operations under your belt you should do just fine! The tricky part is actually putting it in your calculator. Putting in (b.) with no parentheses will most likely get you gibberish, especially if you are someone who presses "enter" after every operation. Make sure to use parentheses just like the ones shown, or at the very least calculate the two different products FIRST and then add them together.

Question (c.) is something you may come across a lot in astronomy – the case where you have exponents (usually 10 raised to something) in the numerator AND denominator. What we will exploit here is that fact that we have a *product* in both the numerator and denominator, i.e., we do not have addition/subtraction in either. This lets us essentially break up a single division operator into multiple. Let's look:

$$\frac{(A \times 10^4)}{(B \times 10^5)} = \frac{A}{B} \times \frac{10^4}{10^5} \ = \ C' \times (10^4) \times (10^{-5}) \ = \ C' \times 10^{-1}$$

Where C'=A/B. Once you get the hang of manipulating exponents, then rearranging problems like this will ultimately make your calculations shorter and much simpler.

Unit Conversion

Unit conversion is very simple once you bury in your brain one simple idea: treat units themselves as numbers (or just *things*) that can cancel out one another when divided. For example, you do this when you are driving all the time. You know how fast you are going on the highway, and you know how far you have to drive. How long, then, will it take for you to get to the place you're headed? In your head you take the distance (in miles) and divide it by your speed (in miles per hour). *Miles* and *hours* are your units here (I'll assume we all know fundamentally what a *unit* is…). Let's write this out:

$$time\ (hour)\ =\ \frac{distance\ (miles)}{speed\ (\frac{miles}{hour})}$$

If we treat these units like any other number, then the *miles* in the numerator and denominator *cancel each other out*, and the *hour* unit jumps up to the numerator (since, of course, we know that from our knowledge of fractions, right?). So, we basically have:

$$time\ (hour)\ =\ \frac{distance}{speed}(hour)$$

and we see that the units (put strategically in parentheses) are the *same on both sides*. Now remember, we said that the *miles* unit only canceled because we had it in both the numerator and denominator. If we measured distance in kilometers and speed in miles per hour, then we wouldn't be able to cancel them out because *miles* and *kilometers* are obviously not the same! HOWEVER, in this case we could do a bit of *dimensional analysis* to make this work! If we have the distance measured in kilometers and the speed measured in miles per hour, and we still want the time in hours, then we can *multiply this by a fraction that equals one* (since it is perfectly legal to multiply both sides by one!) *AND that eliminates the 'kilometers' unit while introducing the 'miles' unit.* This fraction is called a *conversion factor.* Since the only way to eliminate the *kilometers* unit in the numerator is to cancel it out with one in the denominator, this conversion factor must have *kilometers* in the denominator and *miles* in the numerator. Let's write this out:

$$time\ (hour)\ =\ \left(\frac{distance\ (kilometers)}{speed\ (\frac{miles}{hour})}\right)\times\left(\frac{A\ (miles)}{B\ (kilometers)}\right)$$

See that after working out this problem similarly to what we did above, we get the kilometers to cancel, the miles to cancel, and then we are left with hours; just like we wanted. However, now we are left with an extra A/B multiplied by our original "distance/speed" since it is obvious that the actual values of those terms have changed.

So how do we determine the conversion factor? Well, we said it had to be equal to one, which means the numerator and denominator must be the *same*. Now wait a second, I just said that miles don't cancel out with kilometers because they aren't the same – how can this equal one?! Good question – it does so for the same reason that

$$\frac{3\times 4}{6\times 2}\ =\ 1$$

Even though the numbers themselves are different (remember we are thinking of units as being similar to numbers), their *products* of the numerator and denominator separately are the **same**.

So yes, the units themselves are not the same but the values the numerator and denominator represent are **equivalent**. For example, if you run one mile, you have equivalently walked 1.6 kilometers – they are the same distance just expressed differently! Since they are equivalent, if you divide these then you (by definition) get one!

$$\frac{1\ mile}{1.6\ kilometers} = 1$$

HA! This looks exactly like what we are looking for in our dimensional analysis problem! This is the process you must think about when you do these types of problems – you need to think about what kind of fractions you need to introduce to make the units cancel leaving only that which you wish to find (hours in our example above) WHILE making sure that the fraction ultimately equals ‘1’.

Now what we did above can be applied to problem 5 so that you can be confident you are doing your math correctly. Let’s actually work through the first one; (a.). First, I have some amount of *inches* that I want to convert to the equivalent number of *feet*. You can probably do this in your head, but let’s work through this so you can apply it to harder problems. I want to multiply by a fraction with *inches* in the denominator (so that the two cancel) and *feet* in the numerator (so that *feet* becomes the only unit not cancelled out – the unit associated with my final answer).

$$12\ (inches) \times \left(\frac{A\ (feet)}{B\ (inches)}\right) = x\ (feet)$$

Where A, B, and x are unknowns at the moment. Let’s determine this fraction by asking this question: “how many **inches** are in **one foot**?” Well, we know there are 12 inches in one foot – as in 12 inches is *equivalent to* one foot. Thus;

$$\frac{1\ (feet)}{12\ (inches)} = 1$$

Perfect, so we know A and B. Lets plug these in, cancel out the appropriate units, and then figure out our final answer:

$$12\ (\cancel{inches}) \times \left(\frac{1\ (feet)}{12\ (\cancel{inches})}\right) = \frac{12 \times 1}{12}(feet) = 1\ (feet)$$

Thus 12 inches converted to feet is just one foot (as we know). Now how many *miles* is equivalent to 12 *inches*? It should be obvious that this will be a small number; 1 mile is already much bigger than 12 inches, so 12 inches must be some *fraction* of one mile. We can actually build upon the previous question and just add another conversion factor that is also equal to one:

$$12\ (inches) \times \left(\frac{1\ (feet)}{12\ (inches)}\right) \times \left(\frac{A\ (miles)}{B\ (feet)}\right) = x\ (miles)$$

We know (or can google) that 1 mile is equivalent to 5280 feet, so just like before we can define A and B since

$$\frac{1\ (miles)}{5280\ (feet)} = 1$$

Let’s plug all of these in now:

$$12\ (\cancel{inches}) \times \left(\frac{1\ (\cancel{feet})}{12\ (\cancel{inches})}\right) \times \left(\frac{1\ (miles)}{5280\ (\cancel{feet})}\right) = \frac{12 \times 1 \times 1}{12 \times 5280}(miles)$$
$$= 1.89 \times 10^{-4}(miles)$$

Make sure your math matches mine before you move on!

For questions 6, you can probably do this in your head, or you can do a procedure similar to that with the unit conversions above just to make sure you don't make silly mistakes. Questions 6-8 here are just another exercise in doing operations on exponents/numbers in scientific notation!

Algebraic Solutions

The golden rule for solving algebraic equations is that *what you do to one side of the equation you must do to the other*. Let's walk through 9b.).

$$F = G\frac{mM}{R^2}$$

Here we want to solve for R – as in we want R by itself on one side of the equation (NOT R^2, just R). There are many different ways you can go about this, but personally I like to put the variable I am looking for alone on the left-hand side (LHS) of the equation. To do this, I need to *multiply each side of the equation by R^2.*

$$(R^2)F = G\frac{mM}{R^2}(R^2)$$

See that now on the right-hand side (RHS) we have an R^2 in the numerator AND the denominator – thus they *cancel out* (i.e., $R^2/R^2=1$). Thus, the equation above is equivalent to:

$$R^2F = GmM$$

Now we need a way to cancel out F on the LHS – we can divide both sides by F:

$$\left(\frac{1}{F}\right)R^2F = GmM\left(\frac{1}{F}\right)$$

Again, we have F in the numerator AND the denominator on the LHS, so this is now equivalent to:

$$R^2 = G\frac{mM}{F}$$

Remember that I said we want R **by itself**, so we need to get rid of the 'squared' operator. How do we do this? Well from our earlier discussion we know that the opposite operation is the *square root.* In this case let's use exponents in order to get some practical exercise with them. We can take the quantities on both sides of the equations and raise them to the exponent ½:

$$(R^2)^{1/2} = \left(G\frac{mM}{F}\right)^{1/2}$$

Using the exponent rules on the LHS, we can see that we have found R by itself. Thus:

$$R = \left(G\frac{mM}{F}\right)^{1/2} = \sqrt{G\frac{mM}{F}}$$

The remaining questions can be easily solved using the techniques shown here. You just have to apply our discussion on fractional exponents in order to get rid of the exponent 4 in 9c.).

Percent Difference

The acceleration due to gravity can be found with a simple experiment involving a rock and a stopwatch. However, the value we find doing this experiment will most likely not be too close to the same value found in more precise (and expensive) experiments. We need a way to determine how close our measurement is to this *accepted* measurement. How could we do this? Well, first we could see what it means to find the difference between the two values. However, the difference doesn't really tell us much. For example, we could have a measurement in some experiment of "9 units" while the *accepted* value is "10 units." The difference here between the two is 1. Now how about another measurement (for a different experiment) of "95 units" where the *accepted* value is "100 units." The difference between the two is 5 and this difference is obviously larger than that of the first experiment. Does this make my second experiment less "good" than the first? Actually it doesn't – just taking the difference doesn't take into account how large or small our original numbers are (those in our data set), so it is impossible to make simple comparisons like this. The ***percent difference***, however, takes this difference between the *accepted* measurement and our experimental measurement and determines what fraction it is of the entire value of the *accepted* value (and then turns this into a *percentage*). Let's take our first example experiment above; the difference between the *accepted* value and the experimental value is "1 unit." The *accepted* value is "10 units." 1 is 1/10th of the accepted value – or 10%.

Now how about the second example experiment? The difference was "5 units" while the *accepted value* was "100 units." 5 is 1/20th of this accepted value – or 5%. The **percent difference** in this experiment is *lower* than in the first – meaning that the second experiment was MORE PRECISE than the first, even though the difference between the *accepted* and experimental values were greater.

Now lets write out a simple equation that will give you the percent difference given you have an *accepted* value along with your experimental measurement:

$$Percent\ Difference\ (\%) = \left|\frac{accepted\ value - experimental\ value}{accepted\ value}\right| \times 100\ (\%)$$

Let's discuss a few things about this equation:

1) Note that the vertical bars here are **absolute values** – meaning that this is ALWAYS a positive value!! Your percent difference should NEVER have a negative sign in front of it.
2) Multiplying by 100(%) on the right-hand side turns the fraction (in the absolute value) into a percentage. Thus, using this equation, your answer should ALWAYS be followed by a % sign!

Let's go back to our second example experiment above and plug in numbers so you can get a taste of how this equation works. Again, the *accepted* value is "100 units" and the experimental value is "95 units." Plugging in these numbers we get:

$$Percent\ Diff\ (\%) = \left|\frac{100 - 95}{100}\right| \times 100\ (\%) = \left|\frac{5}{100}\right| \times 100(\%) = \frac{5 \times 100}{100}(\%) = 5\%$$

Note that all my numbers were positive so I went ahead and dropped the absolute value sign in the second to last step above. Now you can use this to figure out your last problem in your lab!